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(Re)constructing dimensions

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Abstract: Compactifying a higher-dimensional theory defined in $\mathbb{R}^{1,3+n}$ on an n-dimensional manifold $\mathcal{M}$ results in a spectrum of four-dimensional (bosonic) fields with masses $m_i^2 = \lambda_i$, where $-\lambda_i$ are the eigenvalues of the laplacian on the compact manifold. The question we address in this paper is the inverse: given the masses of the Kaluza-Klein fields in four dimensions, what can we say about the size and shape (i.e. the topology and the metric) of the compact manifold? We present some examples of isospectral manifolds (i.e., different manifolds which give rise to the same Kaluza-Klein mass spectrum). Some of these examples are Ricci-flat, complex and Kähler and so they are isospectral backgrounds for string theory. Utilizing results from finite spectral geometry, we also discuss the accuracy of reconstructing the properties of the compact manifold (e.g., its dimension, volume, and curvature etc) from measuring the masses of only a finite number of Kaluza-Klein modes.

Keywords: Field Theories in Higher Dimensions, Extra Large Dimensions, Superstring Vacua.

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1. Introduction

In recent years, motivated by the brane world scenario [1]–[6], there has been an enormous interest in exploring the experimental signatures of extra dimensions [7]. In the context of string theory, information about the size and shape of the extra dimensions could shed light on the nature of our string vacuum. Indeed, one of the outstanding challenges in connecting string theory with low energy physics is to understand how the enormous degeneracy of string vacua is lifted. Although the underlying mechanism which selects our string vacuum is unknown at present, over the years we have gained a great deal of insights into the physical implications of string theory by exploring compactifications of string and M theory on various manifolds. Since the spectrum of particles and their interactions in the low energy theory are determined by the size and shape of the extra dimensions, we can use phenomenological constraints to focus our attention on the subset of manifolds that are relevant in describing our observed four-dimensional physics. For example, in compactifying the weakly coupled heterotic string on a Calabi-Yau manifold with the standard embedding, one finds that the number of generations of chiral matter is determined by the Euler character whereas the Yukawa couplings are given by the intersection numbers of cycles in the Calabi-Yau space. In brane world models where chiral fermions arise from intersecting branes [8]–[14], the number of chiral generations is again determined by a topological quantity, namely, the intersection number between different D-branes.[1] The basic premise of string phenomenology is to make use of some geometrical/topological insights to construct string models that satisfy some basic physical requirements (e.g., models with three generations, the correct structure of Yukawa couplings and gauge couplings, etc.) from which we can extract their low energy predictions. In addition to the Standard Model-like spectrum of light fields, different compactifications generically give rise to different spectra of Kaluza-Klein (KK) modes, winding modes as well as stringy states. Therefore, if we were able to measure the mass spectrum of these heavy modes (i.e., the size and shape of the extra dimensions), we could further constrain the types of compactifications. This is also the philosophy behind the studies of the experimental signatures of different extra dimension scenarios as the KK spectra are different for large extra dimensions [2], warped compactifications [6], compact hyperbolic manifolds [16], and etc.

The question we address in this paper is the reverse, namely, *given the mass spectrum of the Kaluza-Klein (KK) modes, how well can we reconstruct the extra dimensions?* As we shall see, there exist distinct manifolds which nonetheless have the same set of eigenvalues of the laplacian. Therefore, compactifications on these *isospectral manifolds* result in identical KK mass spectrum. Interestingly, some of the isospectral manifolds are Ricci-flat, complex, and Kähler. Hence they provide isospectral backgrounds not only for point particles but also for the propagation of strings. In reality, we cannot measure the full KK spectrum and so perhaps a physically more relevant question is: how much information about the properties of a compact manifold (e.g., its dimension, volume, and curvature etc) can we extract from measuring the masses of a finite number of Kaluza-Klein modes? Utilizing results from the subject of finite spectral geometry, we obtain some quantitative estimate of the accuracy in reconstructing the geometrical properties (such as the dimension, volume and curvature) of the compact manifold.

The fact that geometrical interpretations are not unique is not a new surprise in string theory – one of the notable examples of such non-uniqueness is mirror symmetry. It has long been conjectured [17, 18] and demonstrated by explicit examples [19, 20] that in string theory there are topologically distinct manifolds which give rise to identical physical models. The equivalence of mirror manifolds is due to the extended nature of strings – roughly speaking, the KK modes and the winding modes are interchanged under mirror symmetry. Compactifications on isospectral manifolds correspond to a simpler form of equivalence — the mass spectra of the KK modes and the winding modes of isospectral manifolds are identical individually. Therefore such equivalence holds even in the point particle ($\alpha' \to 0$) limit. However, we note that unlike mirror manifolds which give rise

[1] In a T-dual picture, these numbers are related to Dirac indexes [15], that again are topological quantities.

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to physical models with identical spectrum as well as interactions, compactifications on isospectral manifolds could differ at the level of interactions and so these manifolds can be disentangled with additional measurements such as the couplings between different KK modes.

An important lesson drawn from the idea of deconstruction [21, 23][2] is that from the mass spectrum of only a finite number of KK modes (the low lying modes), we cannot distinguish between a higher-dimensional theory and a four-dimensional gauge theory which becomes strongly coupled in the infrared. Here, we shall see that even the full KK mass spectrum does not give us sufficient information to reconstruct the compact manifold.

This paper is organized as follows: in section 2 we revisit the general properties of a free scalar in a compact space. This will be our toy model to discuss some general properties of compactifications. Then, in section 3, we analyze the asymptotic expansion of the heat kernel given properties of the manifold that we can read from the spectrum of the laplacian. In section 4, we compute the heat kernel for some simple examples: flat tori and spheres. In particular, we will see how to read off different information about the manifold from these functions. In section 5, we will discuss how the laplacian acting on forms can give more information. Then we will describe some properties of different manifolds with the same spectrum, i.e. isospectral manifolds. In section 7 we will briefly describe how interactions can distinguish between isospectral manifolds. We then discuss the implications of isospectral manifolds to string theory: we will consider some examples of isospectral manifolds where the conformal theory can be constructed, e.g. flat tori. Then we will discuss some properties of general string backgrounds, in particular, how strings will not be able classically to distinguish isospectral manifolds from their spectrum. Finally, in section 9, we will address the question of what can we say with only a finite number of eigenvalues, i.e., the low-lying Kaluza-Klein masses.

2. Scalar field on a compact manifold

Let us take the simplest example: consider a free scalar field living on $\mathbb{R}^{1,3} \times \mathcal{M}$, where we take $\mathcal{M}$ to be a compact manifold without boundary of dimension n:

$$\phi : \mathbb{R}^{1,3} \times \mathcal{M} \longrightarrow R\,. \tag{2.1}$$

From the four dimensional perspective, we get a set of scalar fields with a mass^2 $m_i^2 = \lambda_i$, where $-\lambda_i$ are the eigenvalues of the laplacian operator on the compact riemannian manifold, i.e. a manifold plus a metric on it $(\mathcal{M}, g)$:

$$\Delta_g \phi_i = -\lambda_i \phi_i\,. \tag{2.2}$$

The eigenfunctions, ϕ_i, can be taken to be an orthonormal basis for the harmonic functions on $\mathcal{M}$:

$$\int_{\mathcal{M}} \phi_i \phi_j = \delta_{ij}\,. \tag{2.3}$$

[2]Some similar ideas were considered in [22].

The spectrum of the laplacian operator, i.e. the set of its eigenvalues counted with the corresponding multiplicities $\{\lambda_i\}$, is a discrete sequence of real positive number with the following properties:

- there is a unique zero mode which is just a constant $\phi_0 = 1/\sqrt{\mathrm{Vol}(\mathcal{M})}$;
- the eigenvalues form a numerable, infinite set: $\lambda_i \leq \lambda_{i+1}$;
- the limit $\lim_{i\to\infty} \lambda_i = \infty$;
- the integral of the eigenfunctions in the compact space vanishes except for the zero mode:
$$\int_{\mathcal{M}} \phi_i = \sqrt{\mathrm{Vol}(\mathcal{M})}\delta_{i0}\,; \tag{2.4}$$
- in the limit where $k \to \infty$ the eigenvalues λ_k satisfy the Weyl asymptotic formula:
$$\lim_{k\to\infty} \frac{\lambda_k}{k^{2/n}} = \frac{c_n}{(\mathrm{Vol}(\mathcal{M}))^{2/n}}\,, \tag{2.5}$$
where $c_n = \frac{(2\pi)^2}{\omega_n^{2/n}}$ and ω_n is the volume of the unit ball in R^n. Then at very high masses, $k \to \infty$, the variation of the number of eigenvalues with respect to the variation of the masses is:
$$\delta N = \frac{n\,\mathrm{Vol}(\mathcal{M})}{c_n^{n/2}} m^{n-1}\delta m \tag{2.6}$$
as expected. If the number of dimensions is bigger the number of modes grows very quickly.

Now, the questions that we would like to consider is the following: given the spectrum, what information about the compact manifold can we reproduce? More specifically, what are the geometric properties that are determined by the spectrum? Are there two or more riemannian manifolds with the same spectrum (isospectral)?

3. Heat invariants

Let $\mathcal{M}$ be a smooth compact riemannian manifold without boundary of dimension m. The laplacian on this manifold has a spectrum $\{\lambda_i\}$. Let us define the function (trace of the heat kernel):
$$Z(t) = \sum_i e^{-t\lambda_i}\,. \tag{3.1}$$
Taking the limit $t \to 0$ of the $Z(t)$ we get the Minakshisundaram-Peijel expansion:[3]
$$Z(t) = \frac{1}{(4\pi t)^{m/2}}(a_0 + a_1 t + a_2 t^2 + \cdots)\,, \tag{3.2}$$

[3]Also known in physics as the Schwinger-de Witt expansion (see [44] for a field theory perspective of the heat kernel). The name varies in the literature. Besides Schwinger-de Witt and Minakshisundaram-Peijel, it is sometimes also called the Minakshisundaram-Seeley expansion.

where the a_i are some riemannian invariant functions of the curvature tensor and its covariant derivatives.[4] If the manifold has a boundary there is a term proportional to $t^{1/2}$ in the expansion proportional to the volume of the boundary:

$$a_{1/2} = -\frac{\sqrt{\pi}}{2}\,\mathrm{Vol}(\partial\mathcal{M})\,. \tag{3.3}$$

Let us consider the case without boundary. The first tree terms are:

- The first term is just the volume of the manifold:

$$a_0 = \int_{\mathcal{M}} dx\sqrt{g}\,. \tag{3.4}$$

  This term can be easily obtained by computing the heat kernel as the partition function of a scalar field in a quadratic approximation. That is, this term can be obtained by taking the heat kernel as the partition function of a free scalar field on the manifold and performing the path integral by expanding the fields to quadratic order.

- The second term is the integral of the Ricci scalar:

$$a_1 = \frac{1}{6}\int_{\mathcal{M}} dx\sqrt{g}R\,. \tag{3.5}$$

- The third term is

$$a_2 = \frac{1}{360}\int_{\mathcal{M}} dx\sqrt{g}\left(5R^2 - 2R_{ij}R^{ij} + 2R_{ijkl}R^{ijkl}\right). \tag{3.6}$$

From this expansion we can extract some information that only depends on the masses of the Kaluza-Klein modes:

- The dimension of the manifold can be read off from the pole at $t = 0$. The divergence comes from the growing of the number of modes at high energy. The higher the number of dimensions, the stronger is the divergence.

- From the a_0 term one can obtain the volume. Similarly, in a case of a manifold with boundary one can obtain the volume of the boundary from the $a_{1/2}$ term.

- From the a_1 term we obtain the integral of the Ricci scalar. In the particular case of dimension 2, this integral is related (by Gauss-Bonnet) to the Euler characteristic of the manifold: $a_1 = 2\pi\chi(\mathcal{M})/3$. Then in two dimensions the spectrum of the laplacian encodes topological information — the genus. Put it in another way, two isospectral manifolds in 2 dimensions have the same genus.

- The a_2 term has no interpretation as a topological invariant in any dimension. The same is happening with the higher order terms. However, when wisely combined they can be used to get some topological information. That is the usual trick in some index theorems.[5]

[4] See among others [42, 43].

[5] See, for instance, the book of Gilkey [46].

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In particular, two isospectral manifolds have the same dimension and volume. This asymptotic expansion does not contain all the topological information. For instance, a flat two dimensional torus and a flat Klein bottle have different spectrum but the asymptotic expansion will be the same if they have the same volume.[6] In the asymptotic expansion we loose information that is encoded in terms of the form $e^{l_i/t}$, where l_i are the lengths of closed geodesics. When, $t \to 0$ all these terms vanish like 'non-perturbative' effects. In section 8, this relation between closed geodesics and the heat kernel will be used to discuss the masses of winding states in string theory.

In general, there are some properties that can be obtained from the spectrum, such as the dimension, volume, curvature, genus in dimension 2, etc. These are the audible properties. There are other properties that are not audible, e.g., the first homotopy group [30].

4. Examples

Now we are going to illustrate the behavior of the trace of the heat kernel and with two known examples. The first is a flat torus in any dimension. There the asymptotic behavior is particularly easy to obtain by using the Poisson resummation formula. The second example is the case of round spheres in any dimension where the asymptotics are a little bit more involved.

4.1 Flat tori

The easiest example is a flat n dimensional torus. The spectrum is just:

$$\lambda_m = \sum_{ij} G^{ij} n_i n_j \, , \tag{4.1}$$

where the n_i are integer numbers. The heat trace is:

$$Z(t) = \sum_{n_i \in Z} e^{-t \sum_{ij} G^{ij} n_i n_j} \, . \tag{4.2}$$

By using Poisson resummation and taking the limit $t \to 0$ we obtain the asymptotic expansion:

$$Z(t) \sim \frac{\sqrt{\mathrm{Vol}(\mathcal{M})}}{(4\pi t)^{n/2}} \, . \tag{4.3}$$

The only term that is surviving in this expansion is the first term. Higher order terms vanish due to the dependence on the curvature. In particular, we can deduce for the two dimensional torus that its Euler characteristic is zero.

When Poisson resummation is used we can see all the 'non-perturbative' [7] terms $e^{-l_i/t}$ giving information about the closed geodesics (in string theory they are related to the winding modes). In the limit $t \to 0$ all these terms are exponentially suppressed.

[6] All the a_i vanish except the a_0 term due to the dependence on the curvature.

[7] Non-perturbative in the sense that they cannot be obtained in the small t expansion by perturbations around the zero.

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4.2 Spheres

Let us consider a sphere of dimension n with the usual round metric and radius 1. The eigenvalues of the laplacian are $\lambda_j = j(j+n-1)$. They appear with multiplicity: $deg_j = (\frac{n+j}{n}) - (\frac{n+j-2}{n})$. The heat trace is just:

$$Z(t) = \sum_j deg_j e^{-tj(j+n-1)}. \tag{4.4}$$

The more familiar case of a round S^2 gives:

$$Z(t) = \sum_j (2j+1) e^{-tj(j+1)}. \tag{4.5}$$

The asymptotic expansion $t \to 0$ for the two dimensional sphere is [45]:[8]

$$Z(t) = \frac{1}{t}\frac{e^{t/4}}{\sqrt{\pi t}}\int_0^1 \frac{e^{-x/t}}{\sin(\sqrt{x})}dx = \left(\frac{1}{t} + \frac{1}{3} + \frac{t}{15} + \cdots\right). \tag{4.6}$$

In particular, one can deduce from here that the Euler characteristic is $\chi = 2$.
For a three dimensional round sphere the asymptotic expansion gives [45]:

$$Z(t) = \frac{\mathrm{Vol}(\mathcal{M})}{(4\pi t)^{3/2}}e^t = \frac{\mathrm{Vol}(\mathcal{M})}{(4\pi t)^{3/2}}\left(1 + t + \frac{t^2}{2} + \cdots\right). \tag{4.7}$$

That is very easy to compute by writing:

$$Z(t) = \sum_{j\geq 0}(j+1)^2 e^{-t(j+1)^2+t} = \frac{e^t}{2}\sum_{j\in Z}(j+1)^2 e^{-t(j+1)^2} = -\frac{e^t}{2}\frac{d}{dt}\sum_{j\in Z} e^{-tj^2} \tag{4.8}$$

and using the Poisson resummation formula:

$$Z(t) = -\frac{e^t\sqrt{\pi}}{2}\frac{d}{dt}\left(t^{-1/2}\sum_{j\in Z} e^{-\frac{\pi^2 j^2}{t}}\right). \tag{4.9}$$

That gives the correct asymptotic behavior by taking into account that the volume of a round S^3 of radius 1 is $\mathrm{Vol}(\mathcal{M}) = 2\pi^2$.

5. Higher forms and other fields

In addition to functions, the laplacian can also act on higher p-forms and their corresponding spectra can give us further information about the compact manifold. For instance, the size and the shape of a round sphere in any dimensions is specified by the spectrum of the laplacian on functions together with the spectrum of the laplacian on one forms [33]. Another information than one can get immediately are the Betti numbers, because we know that the number of harmonic p-forms, $\Delta_g C = 0$, is just the Betti number b_p of the manifold (Hodge-de Rham theorem). So in principle, the zero modes alone, i.e. massless

[8] That can be computed by using Mellin transform, for instance.

p-forms in four dimensions, can tell us the Betti numbers of the manifold. So, what is the information we get from higher forms?

The asymptotic expansion can be carried out as in the case of functions, but now the coefficients are different [46, 33, 42]:

$$Z_p(t) = \frac{1}{(4\pi t)^{m/2}}(a_0 + a_1 t + a_2 t^2 + \cdots)\,, \tag{5.1}$$

where:

$$\begin{aligned} a_0 &= c(n,p)\,\mathrm{Vol}(\mathcal{M}) \\ a_1 &= \frac{c_0(n,p)}{6}\int_{\mathcal{M}} dx\sqrt{g}R \\ a_2 &= \frac{1}{360}\int_{\mathcal{M}} dx\left(c_1(n,p)R^2 + c_2(n,p)R_{ij}R^{ij} + c_3(n,p)R_{ijkl}R^{ijkl}\right), \end{aligned} \tag{5.2}$$

where the coefficients can be obtained from the function:

$$c(n,p) = \binom{n}{p} = \frac{n!}{(n-p)!p!}\,. \tag{5.3}$$

The coefficients are:

$$\begin{aligned} c_0(n,p) &= c(n,p) - 6c(n-2,p-1) \\ c_1(n,p) &= 5c(n,p) - 60c(n-2,p-1) + 180c(n-4,p-2) \\ c_2(n,p) &= -2c(n,p) + 180c(n-2,p-1) - 720c(n-4,p-2) \\ c_3(n,p) &= 2c(n,p) - 30c(n-2,p-1) + 180c(n-4,p-2)\,. \end{aligned} \tag{5.4}$$

The non-trivial information on the manifold that is not present in the laplacian on functions comes from the second term a_2 that gives a different linear combination of terms quadratic in the curvature.

By using Poincare duality one can relate the coefficients for p-forms to the coefficients for $(n-p)$-forms: $Z_p(t) = Z_{(n-p)}(t)$. So only half of the forms give us new information.

In particular the laplacian acting on 1-forms gives rise to the following terms:

$$\begin{aligned} a_1 &= \frac{n-6}{6}\int_{\mathcal{M}} dx\sqrt{g}R \\ a_2 &= \frac{n}{360}\int_{\mathcal{M}} dx\left(5R^2 - 2R_{ij}R^{ij} + 2R_{ijkl}R^{ijkl}\right) - \\ &\quad - \frac{1}{12}\int_{\mathcal{M}} dx\sqrt{g}\left(2R^2 - 6R_{ij}R^{ij} + R_{ijkl}R^{ijkl}\right). \end{aligned} \tag{5.5}$$

Thus laplacian on forms can be used to distinguish isospectral manifolds that are not locally isometric. If the manifolds are locally isometric, all the heat kernel asymptotics are the same as they depend only on the curvature.

6. Isospectral manifolds

M. Kac asked the following question in 1966 [25]: Can we hear the shape of a drum? If someone is playing a drum of a particular shape in another room, can we reproduce the shape of this drum by measuring the notes? This is a problem of finding the spectrum of a two-dimensional surface with Dirichlet boundary conditions. The answer is negative and was shown very recently in 1991 by C. Gordon, D. Webb and S. Wolpert [26]. They explicitly constructed a pair of drums with different shape but the same spectrum.

Here we will consider compact manifolds without boundary. The first example of isospectral manifolds was found in 1964 by J. Milnor [27]: two 16 dimensional flat tori.[9] Later, other examples of flat tori were found in dimension four [28, 29]. Isospectrality however is a rare phenomenon: almost all flat tori are uniquely determined by the spectrum and there are at most finitely many tori with the same spectrum [35].

An example was found in 1980 [30] which illustrated that the first homotopy group is not determined by the spectrum. The first examples of continuous isospectral metrics were found in 1984 by [31]. Contrary to continuous isospectral metrics is the notion of spectral rigidity, i.e. there are no deformations of the metric that preserve the spectrum. For instance, all round spheres are spectrally rigid [34].

There are a few riemannian manifolds that are completely determined by their spectra: for instance, all the rounded spheres S^n with $n < 6$ [32]. That can be generalized to all the rounded spheres by considering not only the spectrum on functions but also on one forms [33].

One of the methods used to generate isospectral manifolds is the Sunada method [36]. The basic idea is to start with a riemannian manifold and to quotient it by two different discrete subgroups of the isometries, in such a way that the actions are free and there is a relation between the representations.[10] Then the two orbifolds are isospectral. All the isospectral manifolds constructed in this way have a common riemannian covering and they are always locally isometric. These manifolds can only be distinguished by global properties such as the first homotopy group if the two orbifold groups are not isomorphic. The Sunada method implies not only isospectrality on function but on p-forms [41].

Recently new isospectral manifolds have been constructed that are not locally isometric. The first example dates back to 1991 which involves a pair of isospectral manifolds with boundaries [37]. In 1992 the first example of compact isospectral manifolds without boundary was constructed [38]. The method used in these construction is quite different from the one of Sunada and does not imply local isometry. This method was used to obtain continuous multi-parameter families of isospectral products of spheres (of dimension bigger

[9] As we will see later these are the flat tori associated with the lattices of $E_8 \times E_8$ and Spin(32) algebras. As we all know the partition function is the same for both theories. The distinction comes in the three point function, i.e., the interaction.

[10] More specifically, if there is an unitary isomorphism between the representations of the groups on L^2-functions. Then it is easy to see that this construction gives isospectral manifolds: as the orbifold groups commute with the isometries, the laplacian is invariant and the orbifold groups act on the eigenspaces of the laplacian. The eigenfunctions should be invariant under the group. If the dimension of the invariant states are the same, then the two manifolds are isospectral.

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than three) and tori (of dimension bigger than one) [39], and some simply connected closed isospectral manifolds [40]. However, the heat invariants for the laplacian operator acting on one forms are different.

Recently isospectral non-locally isometric examples have been found in dimension four: $S^2 \times T^2$ and other interesting examples involving continuous isospectral families of metrics on Lie groups [24]. In particular, the heat invariant for the Laplace operator on 1-forms changes. Isospectral metrics have been found for higher dimensional spheres and balls, Riemann surfaces of genus $g > 3$, and etc.

We have seen, in analyzing the heat kernel expansion at $t \to 0$, that two isospectral manifolds have the same dimension (that can be read from the order of the pole at $t = 0$), the same volume (that is the constant in the first term of the expansion), the integral of the Ricci scalar (the second term), and so on. In same cases, such as in two dimensions, one can automatically read off some topological data, e.g., the Euler character.

Furthermore, one can prove for two isospectral manifolds of dimension less than six that if one has constant sectional curvature,[11] the other has constant sectional curvature as well [32, 46]. In particular, if one is isometric to the standard sphere or a real projective space RP^n so is the other.

It is easy to see from the heat kernel expansions that if two manifolds are isospectral for functions, 1 and 2-forms [33] that if one of the manifolds has constant scalar curvature or if it is Einstein so is the other.[12] For instance, we can ask ourselves, is there a manifold isospectral to a K3 that is not another K3? If this is isospectrality in functions, one-forms and two forms we can use this theorem and the fact that the Betti numbers are the same to conclude that if there is such isospectral manifold it should be another K3.

We will see in section 8 same explicit examples of isospectral flat tori in dimension four and sixteen.

7. Field interactions

Previously we have considered free fields on a compact manifold. Can interactions distinguish isospectral manifolds? As the simplest example, let us take a single scalar field with a cubic interaction term:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{g}{3!}\phi^3 \,. \tag{7.1}$$

When reduced to four dimensions the field $\phi = \sum_i a_i \phi_i$ is decomposed into an infinite set of four dimensional scalar fields $a_i : \mathbb{R}^{1,3} \to \mathbb{R}$, where the ϕ_i are the eigenfunctions of the laplacian operator in the compact space. The interaction term in four dimensions looks like:

$$\frac{g}{3!}\sum_{ijk} c_{ijk} a_i a_j a_k \,, \tag{7.2}$$

[11] The sectional curvature is a curvature that is associated to two planes at each point. It can be defined in terms of the Riemann curvature as: $K_{ij} = R_{ijij}/(g_{ii}g_{jj} - g_{ij}^2)$. For instance in a two dimensional manifold is $K = R_{1212}/det(g) = R/2$.

[12] That can be proved very easily by taking the a_2 terms of the expansions for the different forms and observing that the combinations of the curvature are linearly independent.

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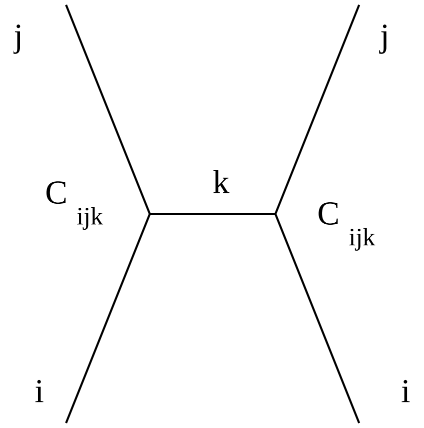

Figure 1: Tree level scattering of Kaluza-Klein modes.

where $c_{ijk} = \int_{\mathcal{M}} \phi_i \phi_j \phi_k$. The interaction with the massless fields $c_{0jk} = \delta_{jk}/\sqrt{\mathrm{Vol}(\mathcal{M})}$ do not distinguish isospectral manifolds. But in general, the c_{ijk} could make the distinction.

Notice that the massive particles do not enter in tree level processes of massless particles due to the $c_{0jk} = \delta_{jk}/\sqrt{\mathrm{Vol}(\mathcal{M})}$. For this toy model, tree level involving massless fields processes are blind to higher modes and in particular they cannot distinguish isospectral manifolds.[13] However the tree level interaction of massive modes depends on c_{ijk}. For instance, one can consider the annihilation of two a_i modes to get two other modes a_j (see figure 1):

$$\mathcal{M}_{ii\to jj} \sim g^2 \sum_k c_{iik} c_{jjk} \frac{1}{p^2 + \lambda_k} . \tag{7.3}$$

By using:

$$\frac{1}{p^2 + \lambda_j} = \int_0^\infty dt e^{-t(p^2+\lambda_j)} \tag{7.4}$$

one can express the amplitude in terms of a heat kernel:

$$\mathcal{M}_{ii\to jj} \sim g^2 \sum_k c_{iik} c_{jjk} \int_0^\infty dt e^{-t(p^2+\lambda_k)} = g^2 \int_0^\infty dt e^{-tp^2} \sum_k c_{iik} c_{jjk} e^{-t\lambda_k} . \tag{7.5}$$

The higher modes enter, for instance, in the correction for the masses of the fields a_i due to all the other modes circulating around a loop (see figure 2):

$$\mathcal{M}_i \sim g^2 \sum_{jk} c_{ijk}^2 \int \frac{dp^4}{(p^2 + \lambda_j)(p^2 + \lambda_k)} . \tag{7.6}$$

One can see that the corrections can be written in terms of the function:

$$Z_i(t, t') = \sum_{jk} c_{ijk}^2 e^{-t\lambda_j} e^{-t'\lambda_k} . \tag{7.7}$$

[13] This is true for our simple model involving only a scalar field. However, for more complicated cases involving gauge fields such as string theory on $E_8 \times E_8$ and $SO(32)$, even the interactions between massless states can distinguish between isospectral manifolds, e.g., the coupling of three gauge bosons.

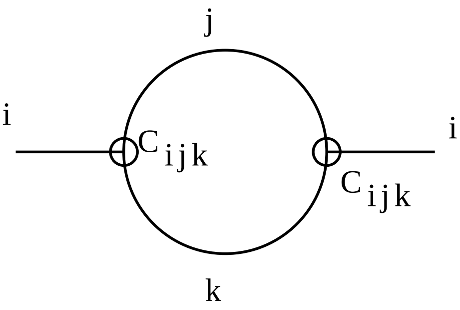

Figure 2: Loop correction to the eigenvalues of the laplacian.

The difference between isospectral manifolds comes in the interaction coefficients c_{ijk} for massive modes. Let us take the corrections to the mass of the lowest state:

$$Z_0(t,t') = \frac{1}{\mathrm{Vol}(\mathcal{M})} \sum_k e^{-(t+t')\lambda_k} = Z(t+t') \,. \tag{7.8}$$

That has the usual UV pole at short distances $t + t' \to 0$:

$$Z_0(t,t') \sim \frac{1}{(4\pi(t+t'))^{n/2}} \,. \tag{7.9}$$

8. Strings on isospectral manifolds

In general we cannot construct the spectrum of string theory for two isospectral manifolds as we do not know the conformal theory description of them. However, as we have seen, there are some flat tori that are isospectral and non-isometric. The first known example is the 16 dimensional tori constructed by Milnor in 1964 [27].[14]. We will analyze the isospectral four dimensional tori of [28, 29] Then we will make some comments about more general backgrounds where we do not know the conformal theory description.

8.1 Flat tori

Flat tori can be constructed by the quotient of flat space $\mathbb{R}^n$ by a lattice. We will use isospectral lattices to describe a pair of lattices that give isospectral tori. In two dimensions the lattice is completely determined by the theta-series. In three dimensions the problem of finding two isospectral lattices is still open. The lowest dimension of two known flat tori that are isospectral is in dimension four [28, 29]. This is the case we will consider. Higher dimensional cases are known in 5, 6, 8, 12 and 16 dimensions.

Let us take the 4-parameter family of isospectral 4-dimensional tori constructed in [29]. Let e_i, $i = 1, \ldots, 4$, a set of orthogonal vectors satisfying the normalization:

$$e_i \cdot e_i = \frac{\alpha_i}{12} \,, \tag{8.1}$$

[14]These lattices are even and self-dual. As one suspects, the two lattices are the $E_8 \times E_8$ and $SO(32)$ lattices [27, 42] that appear in the heterotic strings[59]. $E_8 \times E_8$ and $SO(32)/Z_2$ have the same partition function but interactions (e.g. the algebra that they form) are different. We can only tell the difference between the lattices at the level of interactions.

where $\alpha_i > 0$. The lattices $\Lambda_\pm$ are generated by the vectors (written in the above basis):

$$\begin{aligned} v_1^\pm &= (\pm 3, -1, -1, -1) \\ v_2^\pm &= (1, \pm 3, 1, -1) \\ v_3^\pm &= (1, -1, \pm 3, 1) \\ v_4^\pm &= (1, 1, -1, \pm 3)\,. \end{aligned} \tag{8.2}$$

The two lattices $\Lambda_\pm$ are isospectral.

The metrics on these lattices are:

$$G_{ij}^\pm = v_i^\pm \cdot v_j^\pm\,. \tag{8.3}$$

So the Kaluza-Klein modes, n_i plus the winding modes, m^i, have masses:

$$m^2 = G^{ij} n_i n_j + \frac{G_{ij}}{\alpha'^2} m^i m^j\,. \tag{8.4}$$

The dual lattice of $\Lambda^\pm$ with lengths α_i is $\Lambda^\mp$ with lengths $\alpha_i' = 1/\alpha_i$. One can easily see that they are also isospectral. So string theory cannot distinguish between this pair of isospectral manifolds. In other words, if the lattices of momenta and windings are $(\Lambda^+(\alpha_i), \Lambda^-(1/\alpha_i))$, the isospectral lattices are $(\Lambda^-(\alpha_i), \Lambda^+(1/\alpha_i))$. The operation of mapping isospectral manifolds commutes with T-duality and so the set of dualities on the torus is enlarged.

8.2 Winding modes

In general, we do not have a conformal field theory description of two isospectral manifolds as the flat tori we have seen before, so the question for the general case of two isospectral metrics is still open.

Let us first address the following question: can winding modes distinguish isospectral manifolds? Classically strings can wrap closed geodesic curves. The mass of the states associated with these strings is proportional to the lengths. We can order the closed geodesics β_i by the length $l(\beta_i) \leq l(\beta_{i+1})$. The sequence $l(\beta_1), l(\beta_2), \ldots$ is called the length spectrum. Are the length and the eigenvalue equivalent? In our particular case we are interested in whether the spectrum of geodesic can be deduced from the eigenvalue spectrum.

In general, the two spectra are not equivalent.[15] But there are some cases, e.g., compact Riemann surfaces of genus $g > 1$ where the two spectra are equivalent. For torus (genus 1), the momentum modes determine the windings in flat cases (just Poisson resummation).

But we are not interested in the exact equivalence between the two spectra but in the possible distinction of two isospectral manifolds from the classical winding states. For that, the theorem of Colin de Verdiere on closed geodesics [58] is very useful. Under the generic assumption that the set of closed geodesics are isolated and non-degenerated, it is proved that the eigenvalues of the laplacian determine the spectrum of lengths of closed geodesics. So in a generic case, classical winding states cannot distinguish isospectral manifolds.

[15] See, for instance, [57].

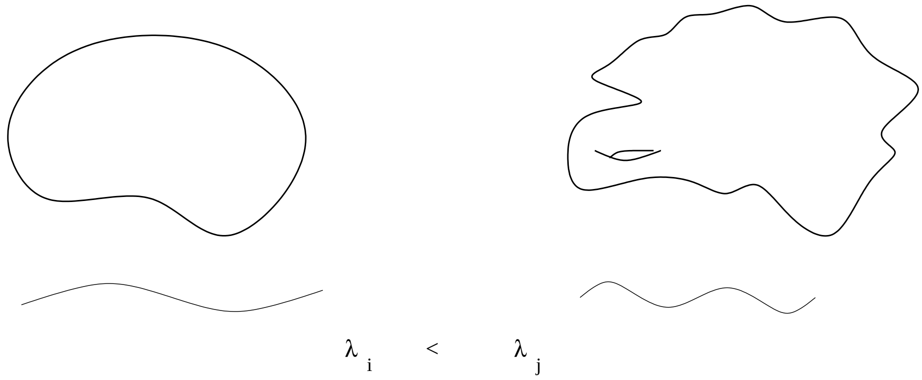

Figure 3: Modes give information about the manifold at some scale λ_i. Higher modes $\lambda_j > \lambda_i$ give more detailed information about the manifold, i.e. information that cannot be resolved by lower modes.

9. What can we extract from a finite number of eigenvalues?

We have seen that even we cannot determine the complete geometry of a manifold by the spectrum one can obtain its dimension, volume, and etc. We can then wonder what kind of information about the manifold can we obtain with only a finite number of eigenvalues (the low-lying ones). There are several ways of understanding this problem.

The first point of view is the one taken in the subject of finite spectral geometry.[16] We can try to estimate the dimension, volume or some constraints on the curvature. What are then the properties of a manifold that we can estimate from a finite number of eigenvalues? How many eigenvalues do we need to estimate them with some given accuracy?

The second viewpoint is more physical. In general, when we are measuring the properties of an object, we use some scale and we define various properties of this object (like the volume, for instance) at this scale.[17] That scale is given in our case by the wavelength of the particle that is propagating in the manifold, which is of the order of the inverse of the mass of this particle. Very massive particles can give very detail information about the structure of the manifold (see figure 3). Now imagine we want to measure the volume of our manifold. Using only the long wavelength modes we get an estimation of it at this scale. Taking into account higher order modes we can refine our measuring. From this point of view it is preferable to talk about the volume at some scale rather than the true volume (whatever it is) since perhaps the latter can never be defined.

The two viewpoints are very different but complementary. The viewpoint of finite spectral geometry assumes a fixed riemannian manifold and by imposing bounds on some properties of the manifold one can obtain information about this manifold with a precision that depends on the bounds. The information on short scales is disregarded because of these bounds. The second viewpoint does not care about what is the exact structure at very small scales, like when we are measuring some gross properties of an object we do not care about the atomic structure of it, we just make a rough estimation at some scale. The

[16] See for instance, the article of J.M. Lee in [47] and [51, 50].

[17] A similar approach has been taken in [49], where the idea is illustrated with flat tori.

second viewpoint also suggests that one needs more eigenvalues to determine topological quantities such as the genus of a Riemann surface in contrast to the volume (as illustrated in figure 3).

As an example take a round S^2 sphere of radius R. As we have seen in the examples, the eigenvalues of the laplacian are $\lambda_j = j(j+1)/R^2$. The number of solutions with an eigenvalue less or equal to λ_j are $N_j = (j+1)^2$ (counting multiplicities). Imagine that from other measurements we find out that there are only two dimensions. From the Weyl's formula we can approximate the volume of the manifold as follows:

$$\mathrm{Vol}(\mathcal{M}) \sim c_2 \frac{N_k}{\lambda_{N_k}} = 4\pi R^2 \left(1 + \frac{1}{k}\right), \tag{9.1}$$

where $c_2 = 4\pi$. So the masses of the fields give the following approximation to the volume of the manifold: $8\pi R^2$, $6\pi R^2$, $16\pi R^2/3$,... Converging to the actual volume of the two sphere $4\pi R^2$. In the more physical interpretation we can say that these values give the volume at different scales, but due to the convergence one can find a well defined limit 4π and higher orders give very little information about the volume.

In what follows we will take the point of view of finite spectral geometry. In general we cannot deduce things like the number of extra dimensions, or the volume of the manifold without assuming something about the curvature of the manifold. That can be seen in a very simple example by J.M. Lee [50]: consider a riemannian manifold $\mathcal{M}$ with some spectrum λ_i. Now take a manifold $\mathcal{M} \times S^1_\epsilon$, where S^1_ϵ is a circle of radius ϵ. The eigenvalues of the laplacian on this new manifold are: $\lambda_i + n^2/\epsilon^2$. For ϵ small enough, the spectrum of the two manifolds agree in the first N eigenvalues. This is an example of a family of finite isospectral manifolds. So in order to get some information about the manifold from a finite part of the spectrum some geometric assumptions should be made. As we will see, some of these assumptions can be that the curvature is bounded.

The basic idea of [48, 51, 50] is to take a formula involving a limit of the eigenvalues (like the Weyl formula or the heat trace expansion) and estimate the accuracy of measuring some properties of the manifold (e.g., the volume) with an error function. Let us take, for instance, the Weyl formula and let us define the error function:

$$E(k) = \left| \frac{\lambda_k}{k^{2/n}} - \frac{c_n}{\mathrm{Vol}(\mathcal{M})} \right|. \tag{9.2}$$

Then one try to estimate $E(k)$, i.e., to show the existence of inequalities of the form $E(k) \leq f(k, c_i)$, where the c_i are some quantities with a definite geometric meaning. For instance, the theorem of Li and Yau [48] for convex euclidean domains of fixed dimension (domains of R^n where every two points are connected by a geodesic in the domain) says that there is a number N such that higher modes $k > N$ satisfy $E(k) < \epsilon$. The number N of eigenvalues needed to measure the volume of the domain depends only on ϵ and the eigenvalues $k < N$.

One can repeat the argument for the heat trace and try to find some bound for the volume of a closed manifold. That is done in [51, 50] by considering manifolds with some bound on the Ricci curvature, sectional curvatures, and so on.

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In particular, we can get some bounds on the number of eigenvalues needed to obtain the dimension. Let us suppose $\mathcal{M}$ to be a connected, compact Riemann manifold without boundary with sectional curvature bounded from above and below $b > R_{sect} > \kappa > 0$, then there is a value N of the low-lying eigenvalues from the spectrum of $\mathcal{M}$ such that the dimension is determined by them.

Another piece of interesting information is the spectral gap, i.e. the first eigenvalue λ_1.[18] For instance, take the classical estimate of Lichnerowicz [53]: if the Ricci curvature is bounded from bellow by $R \geq (n-1)\kappa > 0$, where n is the dimension and κ is some positive real number, then the spectral gap is bigger than $\lambda_1 \geq n\kappa$. This inequality becomes an equality only for the usual sphere [54]. But this information is completely useless if $\kappa = 0$.

More interesting is the result of Li [55] that says that if the Ricci scalar $R \geq 0$ then the spectral gap satisfies $\lambda_1 \geq \pi^2/(2d^2)$ where d is the diameter of the manifold. This limit has been improved [56] to $\lambda_1 \geq \pi^2/d^2$. That tells us that if we assume that the manifold has a metric with non-negative Ricci scalar curvature then we can get an estimation of the diameter from the first eigenvalue: $d \geq \pi/\sqrt{\lambda_1}$.

So, let us summarize the above discussions. In general, by taking the most general riemannian manifolds one cannot say what is the dimension, the volume, and so on. However for reasonable manifolds (avoiding strong curvature, for instance) we can establish some bounds on the dimension and the volume of the manifold. It would be interesting to get other bounds on the Euler characteristic for a Riemann surface from the asymptotic expansion of the heat trace. Also we have seen that interesting bounds on the diameter of the compact space can be obtained by the first eigenvalue, i.e. from the spectral gap, for positive Ricci scalar manifolds.

Another way of understanding this is if we want to obtain same details about the compactification structure from the long wavelengths modes (i.e., the lowest modes in the spectrum), we have to decouple the very energetic ones. Under some conditions these very massive modes do not contribute and we are able to give information about the shape of the manifold. The dimension and the volume become audible quantities.

10. Conclusions

In this paper we have explored many issues surrounding the question of how much information can one obtain from the spectrum (i.e. the masses) of the Kaluza-Klein modes. We have seen that one can reconstruct properties of the manifold such as the number of dimensions, the volume, and in some cases, like in two dimensional manifolds, topological information such as the genus. In reality, however, even if one day we can detect these particles, we will have access to only a finite number of them. In this case we discussed how one can reproduce some information about the manifold if certain assumptions are taken into account. In particular, if we assume that the manifold has a non-negative scalar curvature, then the spectral gap gives us a lower bound on the size of the manifold. In general, the modes at some mass $m = \sqrt{\lambda}$ give us information about the riemannian manifold

[18] See, for instance, [52].

at scale $L \sim 1/\sqrt{\lambda}$. This wavelength puts a limit on the accuracy of our approximation. More detailed properties of the manifold that require a resolution smaller than L remain inaudible.

We have also analyzed some examples of manifolds with the same laplacian spectrum, i.e. isospectral manifolds. These manifolds 'sound' the same and cannot be distinguished by the masses of all the massive modes. In some cases where the conformal field theory for strings propagating in such isospectral backgrounds can be explictly constructed, we have seen that string theory cannot 'hear' the difference between isospectral manifolds. The models relevant for string compactifications that we have analyzed here are just isospectral flat tori. General arguments tell us that the spectrum of the laplacian can predict the spectrum of closed geodesics, i.e. the classical masses of the winding modes.

We also pointed out that although the mass spectrum cannot distinguish between isospectral manifolds, generically interactions can hear the difference between them. Furthermore, interactions also give rise to corrections to the masses of the eigenmodes.

We have taken the 'very' toy model of a scalar field and p-forms on the compact manifold without boundary. Our analysis can be generalized to include fermions, gauge fields with non-trivial bundles, gravity, and etc. It would be interesting to extend our analysis to manifolds with boundaries since in some cases the boundaries could have the physical interpretation as branes.[19] Another interesting direction is to analyze the possible non-commutative deformations of a manifold. Following the work of Connes one can find non-commutative isospectral deformations of a manifold that in principle are spectrally rigid, e.g., a sphere. For instance, it was shown in [60] that any compact riemannian spin manifold whose isometry group has rank $r \geq 2$ admits isospectral deformations to noncommutative geometries.

Many interesting questions remain: Can be hear the holonomy group of a manifold? Can we find two isospectral Calabi-Yau's? How can other fields distinguish isospectral manifolds? We hope to return to some of these issues in the future.

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